

# Strong homotopy algebras of a Kähler manifold

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## Abstract

It is shown that any compact Kähler manifold  $M$  gives canonically rise to two strong homotopy algebras, the first one being associated with the Hodge theory of the de Rham complex and the second one with the Hodge theory of the Dolbeault complex. In these algebras the product of two harmonic differential forms is again harmonic.

If  $M$  happens to be a Calabi-Yau manifold, there exists a third strong homotopy algebra closely related to the Barannikov-Kontsevich extended moduli space of complex structures.

## 1 Introduction

Strong homotopy algebras have been introduced by Stasheff [10] more than 30 years ago in the context of topological  $H$ -spaces. Since that time this remarkable structure (and its cousins) has been found at a number of unexpected places, for example in the string field theory, in the topological conformal field theory, in the Morse theory and in the symplectic Floer theory (see, e.g., [11, 12, 2, 6, 7] and references cited therein).

In this paper we show that in the heart of the classical Hodge theory of Kähler manifolds (see, e.g., [3]) there lies a strong homotopy algebra which “explains” the following imperfect behavior of  $d^*$ -closed differential forms: the wedge product,  $\alpha \wedge \beta$ , of  $d^*$ -closed forms  $\alpha$  and  $\beta$  is *not*  $d^*$ -closed in general. This, of course, can be easily fixed by defining a new product  $\circ$ ,

$$\alpha \circ \beta := [\alpha \wedge \beta]_{\text{Ker} d^*},$$

$[\ ]_{\text{Ker} d^*}$  being the natural projection to the  $\text{Ker} d^*$  constituent, but at a price — the product  $\circ$  fails to be associative. Among the main results of the paper is an observation that  $\circ$  does satisfy the *higher order associativity conditions* thereby unveiling the structure of strong homotopy algebra in the Hodge theory of Kähler manifolds. The point is that we are able to write down *explicitly* all the higher order products in terms of the wedge product, Green function and the Kähler form.

In fact there are at least two strong homotopy algebras associated with a compact Kähler manifold  $M$ . The first one is real and is associated with the Hodge theory of the de Rham complex  $(\Omega^\bullet M, d)$ ; the second one is complex (and is not complexification of the first) and is associated with the Hodge theory of the Dolbeault complex  $(\Omega^{\bullet,\bullet} M, \bar{\partial})$ .

If  $M$  happens to be a Calabi-Yau manifold, there exists a third strong homotopy algebra closely related to the Barannikov-Kontsevich extended moduli space of complex structures [1].

The paper is organized as follows. In Sect. 2 we recall the definition of a strong homotopy algebra. In Sect. 3 we consider specific algebraic data which gives rise to a particular strong homotopy algebra; a remarkable feature of the construction is that it provides *explicit* formulae for all the higher homotopies. In Sect. 4 we consider a compact Kähler manifold and prove the results mentioned above.

## 2 Strong homotopy algebras

A *strong homotopy algebra*, or shortly  $A_\infty$ -*algebra*, is by definition a vector superspace  $V$  equipped with linear maps,

$$\begin{aligned} \mu_k : \quad \otimes^k V &\longrightarrow V \\ v_1 \otimes \dots \otimes v_k &\longrightarrow \mu_k(v_1, \dots, v_k), \quad k \geq 1, \end{aligned}$$

of parity  $\tilde{k} := k \bmod 2\mathbb{Z}$  satisfying, for any  $n \geq 1$  and any  $v_1, \dots, v_n \in V$ , the following *higher order associativity conditions*,

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r \mu_k(v_1, \dots, v_j, \mu_l(v_{j+1}, \dots, v_{j+l}), v_{j+l+1}, \dots, v_n) = 0, \quad (1)$$

where  $r = \tilde{l}(\tilde{v}_1 + \dots + \tilde{v}_j) + \tilde{j}(\tilde{l} - 1) + (\tilde{k} - 1)\tilde{l}$  and  $\tilde{v}$  denotes the parity of  $v \in V$ .

Denoting  $dv_1 := \mu_1(v_1)$  and  $v_1 \circ v_2 := \mu_2(v_1, v_2)$ , one may depict explicitly the first four floors of the infinite tower of higher order associativity conditions as follows

$$n = 1: \quad d^2 = 0,$$

$$n = 2: \quad d(v_1 \circ v_2) = (dv_1) \circ v_2 + (-1)^{\tilde{v}_1} v_1 \circ (dv_2),$$

$$n = 3: \quad v_1 \circ (v_2 \circ v_3) - (v_1 \circ v_2) \circ v_3 = d\mu_3(v_1, v_2, v_3) + \mu_3(dv_1, v_2, v_3) + (-1)^{\tilde{v}_1} \mu_3(v_1, dv_2, v_3) + (-1)^{\tilde{v}_1 + \tilde{v}_2} \mu_3(v_1, v_2, dv_3),$$

$$\begin{aligned} n = 4: \quad & \mu_3(v_1, v_2, v_3) \circ v_4 - \mu_3(v_1 \circ v_2, v_3, v_4) + \mu_3(v_1, v_2 \circ v_3, v_4) - \mu_3(v_1, v_2, v_3 \circ v_4) + (-1)^{\tilde{v}_1} v_1 \circ \mu_3(v_2, v_3, v_4) \\ & = d\mu_4(v_1, v_2, v_3, v_4) - \mu_4(dv_1, v_2, v_3, v_4) - (-1)^{\tilde{v}_1} \mu_4(v_1, dv_2, v_3, v_4) - (-1)^{\tilde{v}_1 + \tilde{v}_2} \mu_4(v_1, v_2, dv_3, v_4) - (-1)^{\tilde{v}_1 + \tilde{v}_2 + \tilde{v}_3} \mu_4(v_1, v_2, v_3, dv_4). \end{aligned}$$

Therefore  $A_\infty$ -algebras with  $\mu_k = 0$  for  $k \geq 3$  are nothing but the differential associative superalgebras with the differential  $\mu_1$  and the associative multiplication defined by  $\mu_2$ . If, furthermore,  $\mu_1 = 0$ , one recovers the usual associative superalgebras.

The notion of  $A_\infty$ -algebra is a very natural extension of the usual concept of associative superalgebra. The following well-known fact [9] may serve as a confirmation of this statement: the moduli space of infinitesimal deformations of an associative superalgebra  $A$  within the class of associative superalgebras is isomorphic to the second Hochschild cohomology group  $\text{Hoch}^2(A, A)$ , while the moduli space of infinitesimal deformations of the same  $A$  within the class of  $A_\infty$ -algebras can be identified with the full Hochschild cohomology  $\text{Hoch}^*(A, A)$ . In this sense  $A_\infty$ -algebra is a “final” concept.

### 3 An explicit construction of an $A_\infty$ -algebra

Let  $(V, d)$  be a differential associative superalgebra, with  $d$  denoting the differential, i.e. an odd linear map  $d : V \longrightarrow V$  satisfying the Leibnitz identity

$$d(v_1 \cdot v_2) = (dv_1) \cdot v_2 + (-1)^{\bar{v}_1} v_1 \cdot (dv_2),$$

for any  $v_1, v_2 \in V$ .

Let  $W$  be a sub complex of  $(V, d)$ , i.e. a vector subspace  $W \subset V$  invariant under  $d$ . Note that we do *not* assume that  $W$  is a subalgebra of  $V$ . Instead we make the following

**Assumption 3.1** *There exists an odd operator*

$$Q : V \longrightarrow V$$

*such that for any  $v \in V$  the element  $(1 - [d, Q])v$  lies in the subspace  $W$ , where  $[ , ]$  is the supercommutator.*

The resulting datum  $(W \subset V, d, Q, \cdot)$  is almost the same as the one considered by Gugenheim and Stasheff in [4] except that we do not assume that the operator

$$P \equiv \text{Id} - [d, Q] : V \longrightarrow W$$

is identity when restricted to  $W$ ; moreover, it may not be even a surjection.

Since  $[d, P] = 0$ , the product

$$v_1 \circ v_2 := (1 - [d, Q])(v_1 \cdot v_2)$$

makes  $(W, d, \circ)$  into a differential superalgebra which is *not* associative in general. Under the additional assumption that  $P|_W = \text{Id}$  it was proved in [4] that  $\circ$  does satisfy the higher associativity conditions. We shall give a new proof of this fact (under weaker assumption 3.1); moreover, we shall be able to compute all the higher homotopies  $\mu_k$  *explicitly* in terms of  $d$ ,  $Q$  and  $\cdot$  only.

First we define a series of linear maps

$$\lambda_n : \otimes^n V \longrightarrow V, \quad n \geq 2,$$

starting with

$$\lambda_2(v_1, v_2) := v_1 \cdot v_2$$

and then recursively, for  $n \geq 3$ ,

$$\begin{aligned} \lambda_n(v_1, \dots, v_n) &:= (-1)^{n-1} [Q\lambda_{n-1}(v_1, \dots, v_{n-1})] \cdot v_n - (-1)^{n\tilde{v}_1} v_1 \cdot [Q\lambda_{n-1}(v_2, \dots, v_n)] \\ &\quad - \sum_{\substack{k+l=n+1 \\ k, l \geq 2}} (-1)^{k+(l-1)(\tilde{v}_1+\dots+\tilde{v}_k)} [Q\lambda_k(v_1, \dots, v_k)] \cdot [Q\lambda_l(v_{k+1}, \dots, v_n)]. \end{aligned} \quad (2)$$

For example,

$$\begin{aligned} \lambda_3(v_1, v_2, v_3) &= [Q\lambda_2(v_1, v_2)] \cdot v_3 - (-1)^{\tilde{v}_1} v_1 \cdot [Q\lambda_2(v_2, v_3)], \\ \lambda_4(v_1, v_2, v_3, v_4) &= -[Q\lambda_3(v_1, v_2, v_3)] \cdot v_4 - (-1)^{\tilde{v}_1+\tilde{v}_2} [Q\lambda_2(v_1, v_2)] \cdot [Q\lambda_2(v_3, v_4)] \\ &\quad - v_1 \cdot [Q\lambda_3(v_2, v_3, v_4)] \\ \lambda_5(v_1, v_2, v_3, v_4, v_5) &= [Q\lambda_4(v_1, v_2, v_3, v_4)] \cdot v_5 + (-1)^{\tilde{v}_1+\tilde{v}_2+\tilde{v}_3} [Q\lambda_3(v_1, v_2, v_3)] \cdot [Q\lambda_2(v_4, v_5)] \\ &\quad - [Q\lambda_2(v_1, v_2)] \cdot [Q\lambda_3(v_3, v_4, v_5)] - (-1)^{\tilde{v}_1} v_1 \cdot [Q\lambda_4(v_2, v_3, v_4, v_5)]. \end{aligned}$$

Setting formally

$$\lambda_1 := -Q^{-1}$$

(which makes sense because in all our formulae below  $\lambda_1$  enters in the combination  $Q\lambda_1 = -\text{Id}$ ), one can rewrite the recursion (2) as follows

$$\lambda_n(v_1, \dots, v_n) = - \sum_{\substack{k+l=n+1 \\ k, l \geq 1}} (-1)^{k+(l-1)(\tilde{v}_1+\dots+\tilde{v}_k)} [Q\lambda_k(v_1, \dots, v_k)] \cdot [Q\lambda_l(v_{k+1}, \dots, v_n)], \quad (3)$$

where now  $n \geq 2$ .

**Lemma 3.2** *The tensors  $\lambda_k$ ,  $k \geq 2$ , satisfy the identities*

$$\Phi_n(v_1, \dots, v_n) \equiv \sum_{\substack{k+l=n+1 \\ k, l \geq 2}} \sum_{j=0}^{k-1} (-1)^r \lambda_k(v_1, \dots, v_j, \lambda_l(v_{j+1}, \dots, v_{j+l}), v_{j+l+1}, \dots, v_n) = 0,$$

$$r = l(\tilde{v}_1 + \dots + \tilde{v}_j) + j(l-1) + (k-1)l,$$

for any  $n \geq 3$  and any  $v_1, \dots, v_n \in V$ .

**Proof.** First we split  $\Phi_n$ ,

$$\begin{aligned} \Phi_n(v_1, \dots, v_n) &= \sum_{\substack{k+l=n+1 \\ k, l \geq 2}} (-1)^{(k-1)l} \lambda_k(\lambda_l(v_1, \dots, v_l), v_{l+1}, \dots, v_n) + \\ &\quad \sum_{\substack{k+l=n+1 \\ k, l \geq 2}} (-1)^{l(\tilde{v}_1+\dots+\tilde{v}_{k-1})+k-1} \lambda_k(v_1, \dots, v_{k-1}, \lambda_l(v_k, \dots, v_n)) + \\ &\quad \sum_{\substack{k+l=n+1 \\ k, l \geq 2}} \sum_{j=1}^{k-2} (-1)^r \lambda_k(v_1, \dots, v_j, \lambda_l(v_{j+1}, \dots, v_{j+l}), v_{j+l+1}, \dots, v_n) = 0. \end{aligned} \quad (4)$$

It is an easy calculation using (3) to check that

$$\begin{aligned}
& - \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} (-1)^{l+k(\tilde{v}_1+\dots+\tilde{v}_l)} \lambda_l(v_1, \dots, v_l) \cdot [Q\lambda_{k-1}(v_{l+1}, \dots, v_n)] \\
& + \sum_{\substack{k+l=n+1 \\ k+l \geq 2}} (-1)^{l(\tilde{v}_1+\dots+\tilde{v}_{k-1})} [Q\lambda_{k-1}(v_1, \dots, v_{k-1})] \cdot \lambda_l(v_k, \dots, v_n) = 0.
\end{aligned}$$

Then the first two sums in (4) reduce, again with the help of (3), to the following expression

$$\begin{aligned}
& - \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{\substack{s+t=k \\ s \geq 2 \\ t \geq 1}} (-1)^p [Q\lambda_s(\lambda_l(v_1, \dots, v_l), v_{l+1}, \dots, v_{s+l-1}))] \cdot [Q\lambda_t(v_{l+s}, \dots, v_n)] \\
& - \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{\substack{s+t=k \\ s \geq 1 \\ t \geq 2}} (-1)^q [Q\lambda_s(v_1, \dots, v_s)] \cdot [Q\lambda_t(v_{s+1}, \dots, v_{k-1}, \lambda_l(v_k, \dots, v_n))],
\end{aligned}$$

where  $p = k(l-1) + s + (t-1)(\tilde{v}_1 + \dots + \tilde{v}_{s+l-1} + l)$  and  $q = l(\tilde{v}_1 + \dots + \tilde{v}_{k-1}) + k - 1 + s + (t-1)(\tilde{v}_1 + \dots + \tilde{v}_s)$ . The third sum in (4) splits into the following two sums

$$\begin{aligned}
& \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{j=1}^{k-2} \sum_{\substack{s+t=k \\ 1 \leq s \leq j \\ t \geq 1}} (-1)^a [Q\lambda_s(v_1, \dots, v_s)] \cdot [Q\lambda_t(v_{s+1}, \dots, v_j, \lambda_l(v_{j+1}, \dots, v_{j+l}), v_{j+l-1}, \dots, v_n)] \\
& - \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{j=1}^{k-2} \sum_{\substack{s+t=k \\ s \geq j+1 \\ t \geq 1}} (-1)^b [Q\lambda_s(v_1, \dots, v_j, \lambda_l(v_{j+1}, \dots, v_{j+l}), v_{j+l+1}, \dots, v_{s+l-1})] \cdot [Q\lambda_t(v_{s+l}, \dots, v_n)],
\end{aligned}$$

where  $a = r + s - 1 + (t-1)(\tilde{v}_1 + \dots + \tilde{v}_s)$  and  $b = r + s + (t-1)(\tilde{v}_1 + \dots + \tilde{v}_{s+l-1} + l)$ .

Substituting these two expressions back into (4) one easily gets the following recursive formula

$$\begin{aligned}
\Phi_n(v_1, \dots, v_n) &= \sum_{\substack{k+l=n \\ k \geq 3 \\ l \geq 1}} (-1)^{(l-1)(\tilde{v}_1+\dots+\tilde{v}_k)+k} [Q\Phi_k(v_1, \dots, v_k)] \cdot [Q\lambda_l(v_{k+1}, \dots, v_n)] \\
&- \sum_{\substack{k+l=n \\ k \geq 1 \\ l \geq 3}} (-1)^{l(\tilde{v}_1+\dots+\tilde{v}_k)} [Q\lambda_k(v_1, \dots, v_k)] \cdot [Q\Phi_l(v_{k+1}, \dots, v_n)],
\end{aligned}$$

where  $n \geq 4$ .

Finally we compute

$$\Phi_3(v_1, v_2, v_3) = (v_1 \cdot v_2) \cdot v_3 - v_1 \cdot (v_2 \cdot v_3) = 0.$$

Thus  $\Phi_n = 0$  for all  $n \geq 3$ .  $\square$

**Lemma 3.3** *The tensors  $\lambda_k$ ,  $k \geq 2$ , satisfy, for any  $n \geq 2$  and any  $v_1, \dots, v_n \in V$ , the identities*

$$\begin{aligned} \Theta_n(v_1, \dots, v_n) &\equiv d\lambda_n(v_1, \dots, v_n) + \sum_{j=0}^{n-1} (-1)^{n-1+\tilde{v}_1+\dots+\tilde{v}_j} \lambda_n(v_1, \dots, v_j, dv_{j+1}, v_{j+2}, \dots, v_n) \\ &\quad - \sum_{\substack{k+l=n+1 \\ k, l \geq 2}} \sum_{j=0}^{k-1} (-1)^r \lambda_k(v_1, \dots, v_j, [d, Q]\lambda_l(v_{j+1}, \dots, v_{j+l}), v_{j+l+1}, \dots, v_n) \\ &= 0 \end{aligned}$$

where  $r = l(\tilde{v}_1 + \dots + \tilde{v}_j) + j(l-1) + (k-1)l$ .

**A scetch of the proof.** Following the same scenario as the one used in the proof of Lemma 3.2 (i.e. studing separately the group of terms with extreme values of the index  $j$ , where some cancellations occur, and applying (3) throughout), one gets the following recursion formula

$$\begin{aligned} \Theta_n(v_1, \dots, v_n) &= \sum_{\substack{k+l=n \\ k \geq 2 \\ l \geq 1}} (-1)^{(l-1)(\tilde{v}_1+\dots+\tilde{v}_k)+k} [Q\Theta_k(v_1, \dots, v_k)] \cdot [Q\lambda_l(v_{k+1}, \dots, v_n)] \\ &\quad - \sum_{\substack{k+l=n \\ k \geq 1 \\ l \geq 2}} (-1)^{l(\tilde{v}_1+\dots+\tilde{v}_k)} [Q\lambda_k(v_1, \dots, v_k)] \cdot [Q\Theta_l(v_{k+1}, \dots, v_n)], \end{aligned}$$

where  $n \geq 3$ .

Next we compute

$$\Theta_2(v_1, v_2) = d(v_1 \cdot v_2) - (dv_1) \cdot v_2 - (-1)^{\tilde{v}_1} v_1 \cdot (dv_2) = 0.$$

Thus  $\Theta_n = 0$  for all  $n \geq 2$ .  $\square$

**Theorem 3.4** *Let  $(V, d)$  be a differential associative superalgebra and let  $(W, d) \subset (V, d)$  be a subcomplex satisfying the Assumption 3.1. Then the linear maps*

$$\mu_k : \otimes^k W \longrightarrow W, \quad k \geq 1,$$

*defined by*

$$\begin{aligned} \mu_1 &:= d, \\ \mu_k &:= (1 - [d, Q])\lambda_k, \quad \text{for } k \geq 2, \end{aligned} \tag{5}$$

*with  $\lambda_k$  being given by (3), satisfy the higher order associativity identitites (1) for all  $n \geq 1$ .*

*Thus there exists on  $W$  a structure of  $A_\infty$ -algebra.*

**Proof.** Denote the l.h.s. of the equation (1) by  $\Psi_n$ . Then  $\Psi_1$  and  $\Psi_2$  vanish because  $(W, d)$  is a subcomplex of  $(V, d)$ . For  $n \geq 3$  one has

$$\Psi_n = (1 - [d, Q])(\Phi_n + \Theta_n).$$

By Lemmas 3.2 and 3.3, the tensors  $\Phi_n$  and  $\Theta_n$  vanish for all  $n \geq 3$ . This completes the proof.  $\square$

The existence part of Theorem 3.4 has been proved by Gugenheim and Stasheff [4] under slightly stronger assumption and using different method. They built up an inductive series of approximations  $\mu_k^{(n)}$ ,  $n \in \mathbb{N}$ , to the higher homotopies  $\mu_k$  and proved its convergence in the limit  $n \rightarrow \infty$  to a genuine  $A_\infty$ -structure<sup>1</sup>.

Finally we note that the higher homotopies  $\mu_n$ ,  $n \geq 2$ , can be written as follows

$$\mu_n(v_1, \dots, v_n) = - \sum_{\substack{k+l=n+1 \\ k, l \geq 1}} (-1)^{k+(l-1)(\tilde{v}_1+\dots+\tilde{v}_k)} [Q\lambda_k(v_1, \dots, v_k)] \circ [Q\lambda_l(v_{k+1}, \dots, v_n)].$$

## 4 $A_\infty$ -algebras of a Kähler manifold

**4.1. Hodge theory: a brief overview.** Let  $M$  be a compact Kähler manifold,  $\Omega^\bullet M$  the algebra of real smooth differential forms on  $M$ ,  $\Omega^{\bullet,\bullet} M$  the algebra of complex differential forms with Hodge gradation, and

$$\partial : \Omega^{\bullet,\bullet} M \longrightarrow \Omega^{\bullet+1,\bullet} M, \quad \bar{\partial} : \Omega^{\bullet,\bullet} M \longrightarrow \Omega^{\bullet,\bullet+1} M,$$

the standard derivations. We also consider two real derivations

$$d = \partial + \bar{\partial} : \Omega^\bullet M \longrightarrow \Omega^{\bullet+1} M, \quad d_c = i(\partial - \bar{\partial}) : \Omega^\bullet M \longrightarrow \Omega^{\bullet+1} M.$$

The Kähler metric  $g$  on  $M$  gives rise to a Hermitian metric on  $\Omega^{\bullet,\bullet}$  defined, say for  $(p, q)$ -forms

$$\alpha = \sum \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} d\bar{z}^{\bar{j}_1} \wedge \dots \wedge d\bar{z}^{\bar{j}_q},$$

and

$$\beta = \sum \beta_{s_1 \dots s_p \bar{t}_1 \dots \bar{t}_q} dz^{s_1} \wedge \dots \wedge dz^{s_p} d\bar{z}^{\bar{t}_1} \wedge \dots \wedge d\bar{z}^{\bar{t}_q},$$

by the integral

$$(\alpha, \beta) := \int_M \left( \sum g^{i_1 \bar{s}_1} \dots g^{i_p \bar{s}_p} g^{t_1 \bar{j}_1} \dots g^{t_q \bar{j}_q} \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} \overline{\beta_{s_1 \dots s_p \bar{t}_1 \dots \bar{t}_q}} \right) \text{vol}_g,$$

where  $g_{i\bar{j}}$  are components of  $g$  in a local holomorphic coordinate chart  $\{z^i\}$ . The scalar product  $(\alpha, \beta)$  of differential forms with different Hodge type is set to be zero.

This scalar product is used to define conjugate operators

$$\partial^* : \Omega^{\bullet,\bullet} M \longrightarrow \Omega^{\bullet-1,\bullet} M, \quad \bar{\partial}^* : \Omega^{\bullet,\bullet} M \longrightarrow \Omega^{\bullet,\bullet-1} M, \quad \Lambda : \Omega^{\bullet,\bullet} M \longrightarrow \Omega^{\bullet-1,\bullet-1} M,$$

by the formulae

$$(\partial\alpha, \beta) = (\alpha, \partial^*\alpha), \quad (\bar{\partial}\alpha, \beta) = (\alpha, \bar{\partial}^*\beta), \quad (\alpha \wedge \omega, \beta) = (\alpha, \Lambda\beta),$$

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<sup>1</sup>After this work was completed L. Johansson drew author's attention to the paper [5] where the limits  $\lim_{n \rightarrow \infty} \mu_k^{(n)}$  have been computed.

where  $\omega$  denotes the Kähler form on  $M$  and  $\alpha, \beta \in \Omega^{\bullet, \bullet} M$  are arbitrary. We shall also need the following real operators

$$d^* = \partial^* + \bar{\partial}^*, \quad d_c^* = -i(\partial^* - \bar{\partial}^*).$$

There are four Laplacians,

$$\begin{aligned} \Delta_{\partial} &= \partial\bar{\partial}^* + \partial^*\bar{\partial}, & \Delta_{\bar{\partial}} &= \bar{\partial}\partial^* + \bar{\partial}^*\partial, \\ \Delta_d &= dd^* + d^*d, & \Delta_{d_c} &= d_c d_c^* + d_c^* d_c. \end{aligned}$$

One of the central results of the Hodge theory says that

$$\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d = \frac{1}{2}\Delta_{d_c}.$$

A differential form  $\alpha \in \Omega^{\bullet, \bullet} M$  is called *harmonic* if  $\Delta_d \alpha = 0$ . The vector space  $\text{Harm}$  of all real (or complex) harmonic forms on  $M$  is finite dimensional and is isomorphic to its de Rham (or Dolbeault) cohomology. There is a natural orthogonal projection

$$\begin{aligned} [\ ]_{\text{Harm}} : \Omega^{\bullet, \bullet} M &\longrightarrow \text{Harm} \\ \alpha &\longrightarrow [\alpha]_{\text{Harm}}. \end{aligned}$$

Note that all the Laplacians  $\Delta$  become isomorphisms when restricted to the orthogonal complement,  $\text{Harm}^{\perp}$ , of the space of harmonic forms.

There are very important for our purposes Hodge identities

$$[\Lambda, \bar{\partial}] = -i\partial^*, \quad [\Lambda, \partial] = i\bar{\partial}^*, \quad [\Lambda, d] = d_c^*, \quad (6)$$

and the Hodge decompositions, for an arbitrary  $\alpha \in \Omega^{\bullet, \bullet} M$ ,

$$\begin{aligned} \alpha &= [\alpha]_{\text{Harm}} + \partial G_{\bar{\partial}} \partial^*(\alpha) + \partial^* G_{\partial} \bar{\partial}(\alpha) \\ \alpha &= [\alpha]_{\text{Harm}} + \bar{\partial} G_{\partial} \bar{\partial}^*(\alpha) + \bar{\partial}^* G_{\bar{\partial}} \partial(\alpha) \\ \alpha &= [\alpha]_{\text{Harm}} + d G_d d^*(\alpha) + d^* G_d d(\alpha) \\ \alpha &= [\alpha]_{\text{Harm}} + d_c G_{d_c} d_c^*(\alpha) + d_c^* G_{d_c} d_c(\alpha), \end{aligned}$$

where  $G : \Omega^{\bullet, \bullet} M \longrightarrow \Omega^{\bullet, \bullet} M$  are the Green operators defined by

$$G_{\partial} |_{\text{Harm}} = 0, \quad G_{\partial} |_{\text{Harm}^{\perp}} = \Delta_{\partial}^{-1},$$

and analogously for all others. A classical fact:

$$G_{\partial} = G_{\bar{\partial}} = 2G_d = 2G_{d_c}.$$

Note that  $\partial G_{\bar{\partial}} = G_{\bar{\partial}} \partial$ ,  $\partial^* G_{\partial} = G_{\partial} \partial^*$ , etc.

For further details about Hodge theory on Kähler manifolds we refer to [3].



**4.2. Real  $A_\infty$ -algebra of a Kähler manifold.** Consider the subcomplex  $(W := \text{Ker}d_c^*, d)$  in the de Rham differential superalgebra  $(\Omega^\bullet M, d)$ . It does satisfy the Assumption 3.1 with

$$Q = d_c G_d \Lambda.$$

Indeed,

$$1 - [d, Q] = 1 - d_c G_d [\Lambda, d] = 1 - d_c G_d d_c^*$$

which, according to the Hodge decomposition, is precisely the projector from  $\Omega^\bullet M$  to its subspace  $\text{Ker}d_c^*$ .

By Theorem 3.4, the space  $\text{Ker}d_c^*$  has canonically the structure of an  $A_\infty$ -algebra with the higher homotopies given explicitly by the formulae (5) and (2). For example,

$$\begin{aligned} \mu_1 &= d, \\ \mu_2(v_1, v_2) &= (1 - d_c G_d d_c^*)(v_1 \wedge v_2) \equiv v_1 \circ v_2, \\ \mu_3(v_1, v_2, v_3) &= [d_c G_d \Lambda(v_1 \wedge v_2)] \circ v_3 - (-1)^{\tilde{v}_1} v_1 \circ [d_c G_d \Lambda(v_2 \wedge v_3)], \\ &\text{etc.} \end{aligned}$$

In this algebra the product of two harmonic forms is obviously harmonic as well. Another interesting observation is that the complex  $(\text{Ker}d_c^*, d)$  is precisely the one which gives rise to a Frobenius manifold structure on the de Rham cohomology of  $M$  [8]. The same comments will apply to our next example.

**4.3. Complex  $A_\infty$ -algebra of a Kähler manifold.** Consider the subcomplex  $(W := \text{Ker}\partial^*, \bar{\partial})$  in the Dolbeault differential superalgebra  $(\Omega^{\bullet, \bullet} M, \bar{\partial})$ . It satisfies the Assumption 3.1 with

$$Q = i\partial G_{\bar{\partial}} \Lambda.$$

Indeed,

$$1 - [d, Q] = 1 - i\partial G_{\bar{\partial}} [\Lambda, \bar{\partial}] = 1 - \partial G \partial^*$$

which, according to the Hodge decomposition, is precisely the projector from  $\Omega^{\bullet, \bullet} M$  to the subspace  $\text{Ker}\partial^*$ .

Therefore, by Theorem 3.4, the space  $\text{Ker}\partial^*$  has canonically the structure of an  $A_\infty$ -algebra and one may explicitly write all the higher homotopies  $\mu_k$  in terms of operators  $\partial$ ,  $\partial^*$ ,  $G_{\bar{\partial}}$  and  $\Lambda$ .

**4.4.  $A_\infty$ -algebra of a Calabi-Yau manifold.** A compact Kähler manifold  $M$  is called Calabi-Yau if  $c_1(M) = 0$  or, equivalently, if it admits a nowhere vanishing holomorphic section  $\Omega$  of the canonical line bundle of  $M$ .

Consider a differential associative superalgebra

$$V := \Lambda^\bullet TM \otimes \Lambda^\bullet \overline{\Omega^1 M}$$

with the differential  $\bar{\partial}$  and the multiplication  $\cdot$  being given by the ordinary wedge products on  $\Lambda^p TM$  and  $\Lambda^q \overline{\Omega^1 M}$ . It was first considered by Barannikov and Konsevich [1] in the context of mirror symmetry.

The holomorphic volume form defines an isomorphism

$$\begin{aligned} \Omega : \Lambda^{\bullet} TM \otimes \Lambda^{\bullet} \overline{\Omega^1 M} &\longrightarrow \Lambda^{\dim M - \bullet} \Omega^1 M \otimes \Lambda^{\bullet} \overline{\Omega^1 M} \\ \alpha \otimes \beta &\longrightarrow (\alpha \lrcorner \Omega) \otimes \beta. \end{aligned}$$

Then for any linear operator

$$s : \Omega^{\bullet, \bullet} M \longrightarrow \Omega^{\bullet, \bullet} M$$

there is associated a linear operator

$$\hat{s} : V \longrightarrow V$$

defined as the composition:

$$\hat{s} : V \xrightarrow{\Omega} \Omega^{\bullet, \bullet} M \xrightarrow{s} \Omega^{\bullet, \bullet} M \xrightarrow{\Omega^{-1}} V.$$

In this way one obtains the operators  $\hat{\partial}$ ,  $\hat{\partial}^*$ ,  $\hat{G}_{\partial}$ , etc. Note that  $\hat{\bar{\partial}} = \bar{\partial}$ . Note also that  $\hat{\partial}$  is not a derivation of  $(V, \cdot)$ .

Consider the following subcomplex

$$(W := \text{Ker} \hat{\partial}, \bar{\partial})$$

in the Barannikov-Kontsevich differential superalgebra  $(V, \bar{\partial})$ . It does satisfy the Assumption 3.1 with

$$Q = -i\hat{\Lambda}\hat{G}_{\partial}\hat{\partial}.$$

Indeed,

$$1 - [d, Q] = 1 - i[\widehat{\Lambda, \bar{\partial}}]\hat{G}_{\partial}\hat{\partial} = 1 - \hat{\partial}^*\hat{G}_{\partial}\hat{\partial},$$

which, according to the Hodge decomposition, is precisely the projector from  $V$  to the subspace  $\text{Ker} \hat{\partial}$ .

Therefore, by Theorem 3.4, the space  $\text{Ker} \hat{\partial}$  has canonically the structure of an  $A_{\infty}$ -algebra and one may explicitly write down all the higher homotopies  $\mu_k$  as explained in Sect.3. It is again a remarkable fact that the same complex  $(\text{Ker} \hat{\partial}, \bar{\partial})$  admits not only an  $A_{\infty}$  but also a very special Gerstenhaber-Batalin-Vilkovski structure which makes the extended moduli space of complex structures on  $M$  into a Frobenius manifold [1].

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